Notes on Perfect Reconstruction Filter Banks (II)

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Abstract

In these brief notes, we review the perfect reconstruction filter banks and in particular, will try answer some of the questions such as: given a random sequence is it always possible to construct bi-orthogonal, perfect reconstruction filter banks? Is the dual of a basis function unique?

1 Introduction

Previously, we have mentioned the conditions required for **perfect reconstruction** filter banks using the Smith-Barnwell view [1]. In these notes we revisit the **perfect reconstruction** problem from the point of view of bi-orthogonal filter banks, of which the orthogonal ones (Smith-Barnwell) are a subset of.

2 Perfect Reconstruction Filter Banks

For a **perfect reconstruction** filter bank, the frequency response is

$$\begin{aligned}\hat{X}(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\omega})H_0(e^{j\omega}) + X(-e^{j\omega})H_0(-e^{j\omega})\right]G_0(e^{j\omega}) + \\ &\quad \frac{1}{2}\left[X(e^{j\omega})H_1(e^{j\omega}) + X(-e^{j\omega})H_1(-e^{j\omega})\right]G_1(e^{j\omega})\end{aligned}$$

re-arranging we have

$$\begin{aligned}\hat{X}(e^{j\omega}) &= \frac{1}{2}\left[H_0(e^{j\omega})G_0(e^{j\omega}) + H_1(e^{j\omega})G_1(e^{j\omega})\right]X(e^{j\omega}) + \\ &\quad \frac{1}{2}\left[H_0(-e^{j\omega})G_0(e^{j\omega}) + H_1(-e^{j\omega})G_1(e^{j\omega})\right]X(-e^{j\omega})\end{aligned}$$

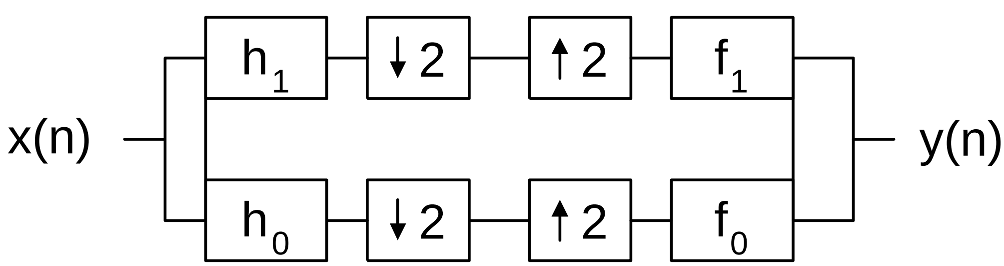

Figure 1: Filter bank - same notation as Smith-Barnwell [1]. Decomposition filters are $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$; reconstruction filters are $G_0(e^{j\omega})$ and $G_1(e^{j\omega})$. (Instead of f there should be g.)

written in matrix format we have

$$\hat{X}(e^{j\omega}) = \frac{1}{2}\begin{bmatrix} H_0(e^{j\omega}) & H_1(e^{j\omega}) \\ H_0(-e^{j\omega}) & H_1(-e^{j\omega}) \end{bmatrix}\begin{bmatrix} G_0(e^{j\omega}) \\ G_1(e^{j\omega}) \end{bmatrix}\begin{bmatrix} X(e^{j\omega}) \\ X(-e^{j\omega}) \end{bmatrix} \tag{1}$$

Assuming that **perfect reconstruction** allows only for a shift (although we could also allow for a scaling factor, as in [1]) we have the following theorem by Vetterli [2].

Theorem 1 *Using the filter bank of Fig. 1* ***perfect reconstruction*** *is achieved if and only if*

$$H_0(e^{j\omega})G_0(e^{j\omega}) + H_1(e^{j\omega})G_1(e^{j\omega}) = 2e^{-jwL} \tag{2}$$

and

$$H_0(-e^{j\omega})G_0(e^{j\omega}) + H_1(-e^{j\omega})G_1(e^{j\omega}) = 0 \tag{3}$$

Proof. ($\Rightarrow$) This is immediately clear from (1). If $\hat{X}(e^{j\omega})$ is an L delay of $X(e^{j\omega})$ then $\hat{X}(e^{j\omega}) = X(e^{j\omega})e^{-jwL}$ and (2), (3) must hold by contradiction. ($\Leftarrow$) Obvious from (1). Q.E.D.

Theorem 1 states that **perfect reconstruction** is achieved if and only if

$$\begin{bmatrix} H_0(e^{j\omega}) & H_1(e^{j\omega}) \\ H_0(-e^{j\omega}) & H_1(-e^{j\omega}) \end{bmatrix}\begin{bmatrix} G_0(e^{j\omega}) \\ G_1(e^{j\omega}) \end{bmatrix} = \begin{bmatrix} 2e^{-jwL} \\ 0 \end{bmatrix} \tag{4}$$

Let M be the 2 by 2 matrix in (4) and let $D(e^{j\omega})$ be the determinant of M. Then we have the following corollary.

Corollary 1 *If* ***perfect reconstruction*** *is achieved then the determinant of matrix M is non-zero.*

Proof. The only way that equation (4) has a solution is if $\left[2e^{-jwL}, 0\right]^T$ is in the range of matrix M. If the determinant is zero, then the range of M is spanned by only one column of M. Clearly, neither of the columns can have the same range as $\left[2e^{-jwL}, 0\right]^T$. So the determinant of M is non-zero. Q.E.D.

We are now ready to introduce our second theorem and the one that is probably the most general in terms of telling us how to build **perfect reconstruction** filter banks.

Theorem 2 *Using the filter bank of Fig. 1* ***perfect reconstruction*** *(with even L) is achieved if and only if the following two sets of equations hold*

$$\begin{cases} G_0(e^{j\omega}) &= \frac{2e^{-jwL}}{E(e^{j\omega})}H_1(-e^{j\omega}) \\ G_1(e^{j\omega}) &= \frac{-2e^{-jwL}}{E(e^{j\omega})}H_0(-e^{j\omega}) \end{cases} \tag{5}$$

and

$$H_0(e^{j\omega})G_0(e^{j\omega}) + H_0(-e^{j\omega})G_0(-e^{j\omega}) = 2e^{-jwL} \tag{6}$$

for some non-zero function $E(e^{j\omega})$ such that $-E(e^{j\omega}) = E(-e^{j\omega})$.

Proof.($\Rightarrow$) To begin with, assume that we have **perfect reconstruction** . By Corollary 1 that means that $D(e^{j\omega}) \neq 0$. From (4) we have

$$\begin{aligned} \begin{bmatrix} G_0(e^{j\omega}) \\ G_1(e^{j\omega}) \end{bmatrix} &= \frac{1}{D(e^{j\omega})}\begin{bmatrix} H_1(-e^{j\omega}) & -H_1(e^{j\omega}) \\ -H_0(-e^{j\omega}) & H_0(e^{j\omega}) \end{bmatrix}\begin{bmatrix} 2e^{-jwL} \\ 0 \end{bmatrix} \\ &= \frac{2e^{-jwL}}{D(e^{j\omega})}\begin{bmatrix} H_1(-e^{j\omega}) \\ -H_0(-e^{j\omega}) \end{bmatrix} \end{aligned} \tag{7}$$

Moreover, with $D(e^{j\omega}) = H_0(e^{j\omega})H_1(-e^{j\omega}) - H_0(-e^{j\omega})H_1(e^{j\omega})$ we have that $-D(e^{j\omega}) = D(-e^{j\omega})$. Let $E(e^{j\omega}) = D(e^{j\omega})$ and this proves (5). To show (6), from (5) we have

$$\begin{aligned} H_1(-e^{j\omega}) &= \frac{E(e^{j\omega})}{2e^{-jwL}}G_0(e^{j\omega}) \\ H_1(e^{j\omega}) &= \frac{E(-e^{j\omega})}{2(-1)^L e^{-jwL}}G_0(-e^{j\omega}) \\ &= \frac{-E(e^{j\omega})}{2(-1)^L e^{-jwL}}G_0(-e^{j\omega}) \end{aligned} \tag{8}$$

The last equation is true since $E(-e^{j\omega}) = -E(e^{j\omega})$. From (8) we have

$$\begin{aligned} H_1(e^{j\omega})G_1(e^{j\omega}) &= \frac{-E(e^{j\omega})}{2(-1)^L e^{-jwL}}G_0(-e^{j\omega})\frac{-2e^{-jwL}}{E(e^{j\omega})}H_0(-e^{j\omega}) \\ &= G_0(-e^{j\omega})H_0(-e^{j\omega}). \end{aligned} \tag{9}$$

(Remember that L is even.) This together with (2) gives us (6)

$$\begin{aligned} 2e^{-jwL} &= H_0(e^{j\omega})G_0(e^{j\omega}) + H_1(e^{j\omega})G_1(e^{j\omega}) \\ &= H_0(e^{j\omega})G_0(e^{j\omega}) + G_0(-e^{j\omega})H_0(-e^{j\omega}) \end{aligned} \tag{10}$$

($\Leftarrow$) To go the other direction, we assume that (5) and (6) hold and show that equations (2) and (3) are true. If (5) is true then (9) holds and together with (6) we have (2), by working our way in the opposite direction in (10). If (5) is true, then in (5) the right side of the first equation in times the left side in the second equation is equal to the left side in the first equation times the right side in the second equation

$$\frac{-2e^{-jwL}}{E(e^{j\omega})}H_0(-e^{j\omega})G_0(e^{j\omega}) = \frac{2e^{-jwL}}{E(e^{j\omega})}H_1(-e^{j\omega})G_1(e^{j\omega})$$

In particular

$$H_0(-e^{j\omega})G_0(e^{j\omega}) + H_1(-e^{j\omega})G_1(e^{j\omega}) = 0$$

which is equation (3). By Theorem 1 we have **perfect reconstruction** . Q.E.D.

Let's look in more details at what Theorem 2 says and how we could use it to build perfect reconstruction filter banks. In particular, it gives us a method of building **perfect reconstruction** filter banks.

Algorithm for building perfect reconstruction filter banks:

1. Find filters $H_0(e^{j\omega})$ and $G_0(e^{j\omega})$ such that (6) is satisfied. More on this later, when we discuss how to go about doing this.
2. Find *any* function $E(e^{j\omega})$ such that $-E(e^{j\omega}) = E(-e^{j\omega})$. To make our life a little simpler, we let $E(e^{j\omega}) = 2e^{-jw(L-1)}$, where L is even. Note also, that with this choice of $E(e^{j\omega})$, filters $H_1(e^{j\omega})$ and $G_1(e^{j\omega})$ are FIR if $H_0(e^{j\omega})$ and $G_0(e^{j\omega})$ are FIR.
3. With the above choice of $E(e^{j\omega})$, using (5) we now have

$$\begin{cases} G_0(e^{j\omega}) &= e^{-j\omega}H_1(-e^{j\omega}) \\ G_1(e^{j\omega}) &= -e^{-j\omega}H_0(-e^{j\omega}) \end{cases}$$

Or, assuming that we have the filters $H_0(e^{j\omega})$ and $G_0(e^{j\omega})$ from first step, we obtain filters $H_1(e^{j\omega})$ and $G_1(e^{j\omega})$ using

$$\begin{cases} H_1(e^{j\omega}) &= -e^{j\omega}G_0(-e^{j\omega}) \\ G_1(e^{j\omega}) &= -e^{-j\omega}H_0(-e^{j\omega}) \end{cases} \tag{11}$$

The critical part of this algorithm is the first step. If we require that $G_0(e^{j\omega}) = H_0(e^{-j\omega})$, (6) becomes

$$|H_0(e^{j\omega})|^2 + |H_0(-e^{j\omega})|^2 = 2e^{-jwL}$$

which is the condition for Smith-Barnwell conjugate mirror filters [1]. The above condition also implies that $H_0(e^{j\omega})$ and $G_0(e^{j\omega})$ are of the same length. Let's look at the case when we don't require $G_0(e^{j\omega}) = H_0(e^{-j\omega})$. Let $F_0(e^{j\omega}) = H_0(e^{j\omega})G_0(e^{j\omega})$. Equation (6) becomes

$$F_0(e^{j\omega}) + F_0(-e^{j\omega}) = 2e^{-jwL} \tag{12}$$

If we let $L = 0$, then (12) implies that $F_0(e^{j\omega})$ must be a Nyquist sequence ($F_0(e^{j\omega})$ and $F_0(-e^{j\omega})$ have the same values at even indexes and opposite values at odd indexes. For their sum to be a constant at index zero, it means that the values of h_0 must be zero at the even indexes).

3 Examples

To build **perfect reconstruction** filter banks all we need is to find two sequences $H_0(e^{j\omega})$ and $G_0(e^{j\omega})$ such that their product is a Nyquist sequence. This observation allows us to partly answer the question of whether given $H_0(e^{j\omega})$ there is a unique $G_0(e^{j\omega})$ that would give **perfect reconstruction** . I believe that the answer is no. That is, given $H_0(e^{j\omega})$, there is more than one $G_0(e^{j\omega})$ that makes $H_0(e^{j\omega})G_0(e^{j\omega})$ a Nyquist sequence.

As a small example, let $H_0(e^{j\omega}) = ae^{jw}+1+be^{-jw}$. Then both $G_0(e^{j\omega}) = 1$ and $G_0(e^{j\omega}) = \frac{1}{a}e^{-jw} + \frac{-b}{a}e^{-2jw}$ form **perfect reconstruction** filter banks. I believe that this is true for other filters as well. To prove that, I would have to show how given FIR $H_0(e^{j\omega})$ we can generate an FIR $G_0(e^{j\omega})$ such that $H_0(e^{j\omega})G_0(e^{j\omega})$ is Nyquist. I'm not sure how to show this for a general case.

Here is a Matlab program that will take in a Nyquist sequence $F_0(e^{j\omega})$ and the roots of $H_0(e^{j\omega})$ and will return the filters $G_0(e^{j\omega}), H_1(e^{j\omega})$ and $G_1(e^{j\omega})$ such that we obtain a **perfect reconstruction** filter bank.

```
function [H0, H1, G0, G1, H0_, H1_, G0_]=makefilters(F0, H0roots)

% this function generates biorthogonal/orthogonal filters
% to be used in a perfect reconstruction filter bank.
% (decimation by 2).
%
% INPUT:  F0 - must be a nyquist sequence.
%              make sure that F0(0) is at an even time index,
%              otherwise (5) would have a minus instead of plus.
%              (EVEN time index is even when we assume that first entry
%               corresponds to the sample at time zero. In this example
%              1 is at odd time index: [0 1].  And this is at an even
%              [1 0].)
%         H0roots - roots of H0.  Make sure that
%              the roots are coming in conjugate pairs!
% OUTPUT: Perfect reconstruction filters.
%
% Written by: Darian Muresan
% Last Modified: April, 14, 2001

% first step is to find the roots of F0.
rt=roots(F0);
% since we have real polynomials.  The roots for
```

```
% H0 must come in conjugate pairs - check that as a user
if nargin<2,
  H0=F0;
  G0=1;
else
  H0=poly(rt(H0roots));
  G0=deconv(F0,H0);
end

% Get H0(w+pi) and G0(w+pi)
H0_=H0; H0_(2:2:end)=-H0(2:2:end);
G0_=G0; G0_(2:2:end)=-G0(2:2:end);


% get H1(w) and G1(w) via (10)
% all of them will have their non-zero indexes lined up;
% don't forget to fix that.
H1=-G0_;
G1=-H0_;
% Fixing the index problem
% The first index of H1 is at -1.
% That means that the we shift H0 and G0
% to the right by one and G1 is actually shifted by two
G0=[0 G0];
H0=[0 H0];
G1=[0 0 G1];
% don't forget to also shift (-w) versions
H0_=[0 H0_];
G0_=[0 G0_];
H1_=H1; H1_(1:2:end)=-H1_(1:2:end);


[H0, H1, G0, G1]=NormLength(H0, H1, G0, G1);

% Return filters of even lengths, in order to
% fix the problem with mydwt.m and myidwt.m
function [H0, H1, G0, G1]=NormLength(h0, h1, g0, g1);

nl=max([length(h0), length(h1), length(g0), length(g1)]);
nl=2*ceil(nl/2);

H0=zeros(1,nl);H0(1:length(h0))=h0;
H1=zeros(1,nl);H1(1:length(h1))=h1;
G0=zeros(1,nl);G0(1:length(g0))=g0;
G1=zeros(1,nl);G1(1:length(g1))=g1;
```

References

[1] Mark J. T. Smith and Thomas P. Barnwell *Exact Reconstruction Techniques for Tree-Structured Subband Coders* IEEE Transactions on Acoustics, Speech, and Signal Processing, Vol. ASSP-34, N0. 3,

pp: 434-440, June 1986.

[2] Stephane Mallat, "A Wavelet Tour of Signal Processing," *Academic Press,* 1998.