ORTHOGONAL SUBSPACE DECOMPOSITION OF PERIODIC SIGNALS

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ABSTRACT

The detection and estimation of machine vibration multi-periodic signals of unknown frequencies in white Gaussian noise is investigated. New estimates for the sub-signals (signals making up the received signal) and their periods are derived using an orthogonal subspace decomposition approach.

1. INTRODUCTION

The analysis of machine vibrations has proven to be a valuable application of signal processing. A variety of well-known techniques used in this area, [1, 2], require that good period and periodic sub-signal estimates can be made. Our aim is to develop a method for periodic sub-signal estimation.

Previous work in the signal estimation area can be found in [3, 4, 5, 6, 7, 8, 9]. Our work extends the results of [3] to multiple period estimation. Our approach is to generate orthogonal subspaces that correspond to periods ranging from 1, to the maximum expected sub-period (P_{max}) of our signal R. Estimates of the sub-signals and their energy are obtained by taking orthogonal projections of R onto these different orthogonal subspaces. We will first analyze the one period and two period estimation cases, then we will generalize the results to multiple period estimation. Finally, using a vibration signal recorded from a General Motors gear box (Fig. 1), we apply our techniques to real data vibration signals.

2. SYNCHRONOUS SAMPLING

In [3], a maximum likelihood pitch estimation method was presented. One difficulty with almost any period estimation method is that the period of the analog signal does not contain an integer number of samples. In other words, the sampled data is never really periodic.

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Figure 1: General Motors gear box used for our real time synchronous sampling. Sampling is synchronized with the position of the large wheel.

Our approach is to force an integer number of samples per period by doing synchronous sampling. Instead of sampling our vibration signal using a fixed frequency clock, our sampling is synchronized with the position of the gears (Fig. 1). Assuming that the periodic sub-signals are generated by different gears in the gear box, a synchronous sampling will provide an integer number of samples per periodic sub-signals.

3. SINGLE PERIOD ESTIMATION

We will briefly review the results in [3] for the single period estimation case. Let $S = \{s_0, \ldots, s_{K_0-1}\}$ be a periodic repetition of the length P sequences $Q = \{q_0, \ldots, q_{P-1}\}$. The received signal $R = \{r_0, \ldots, r_{K_0-1}\}$, of length K_0 (K_0 is a multiple of P) then consists of S plus white, zero mean Gaussian noise $N \sim \mathcal{N}(0, \sigma^2)$

$$R = S + N$$

For any specific period P, an orthonormal basis set for the subspace of all periodic signals of period P is

$$\{\psi_k\} = \sqrt{\frac{1}{M}}\ \delta_k$$

where $k = 0, \ldots, P-1$, $M = K_0/P$ and δ_k a $K_0 \times 1$ vector with i^{th} entry

$$\delta_k(i) = \begin{cases} 1 & i = k + lM, \text{ for integer } l \\ 0 & \text{else} \end{cases}$$

Let Ψ^P be the orthonormal matrix having $\psi_0, \ldots, \psi_{P-1}$ as column vectors

$$[\Psi^P]_{K_0 \times P} = \sqrt{\frac{1}{M}} \begin{bmatrix} 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 1 \end{bmatrix}^T$$

and $\mathcal{R}(\Psi^P)$ be the range of Ψ^P. Then, $\mathcal{R}(\Psi^P)$ is the subspace of signals of period P and any other period $\bar{P}$, for which $\bar{P}$ is a factor of P. For single period estimation, the Maximum Likelihood (ML) estimate minimizes the two norm of the distance between R and $\mathcal{R}(\Psi^P)$ [3]. The ML estimate minimizes

$$\|R - \hat{S}\|^2 = \sum_{k=0}^{K_0-1} (r_k - \hat{s}_k)^2 \qquad (1)$$

where $\hat{S}$ is the projection of R onto $\mathcal{R}(\Psi^P)$, the subspace corresponding to signals of period P (i.e. $\hat{s}_k = \hat{q}_{k \bmod P}$ and $\hat{q}_k = (1/M)\sum_{l=0}^{M-1} r_{k+lP} =< R, \psi_k >$). Minimizing (1) is equivalent to maximizing the square of the 2-norm of $\hat{S}$

$$\|\hat{S}\|^2 = \sum_{k=0}^{P-1} < R, \psi_k >^2 \qquad (2)$$

With $\phi_R(k)$, the autocorrelation function of R, defined as

$$\phi_R(k) = \sum_{j=0}^{K_0-1-k} r_j r_{j+k}$$

it is shown in [3] that

$$\|\hat{S}\|^2 = \frac{P}{K_0}\left[\phi_R(0) + 2\sum_{l=1}^{M-1} \phi_R(lP)\right] \qquad (3)$$

The first term in (3), $\frac{P}{K_0}\phi_R(0)$, grows linearly with P. In order to eliminate some of the bias towards larger periods, in [3], it was eliminated. The proposed period estimate in [3] was

$$\hat{P} = \arg\max_P \left\{ g_{(P,R)} = \frac{2P}{K_0} \sum_{l=1}^{M-1} \phi_R(lP) \right\} \qquad (4)$$

and the signal estimates $\hat{q}_i$ for $i = 0, \ldots, \hat{P}-1$ ($\hat{s}_k = \hat{q}_{k \bmod \hat{P}}$) were

$$\hat{q}_k = \frac{1}{M}\sum_{l=0}^{M-1} r_{k+lP}$$

4. EXACTLY PERIODIC SIGNALS

Let us see what happens when we apply the above algorithm to a simple signal of length 12. Let

$$R = [1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2]$$

Calculating the energy of the projections of R onto the corresponding subspaces, $\mathcal{R}(\Psi^k)$, using equation (3), we obtain the graph of Fig. 2.

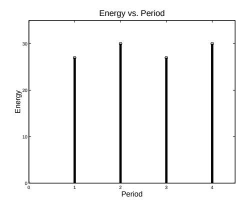

Figure 2: Plot of the energy of the projections of R onto the subspace $\mathcal{R}(\Psi^1), \ldots, \mathcal{R}(\Psi^4)$.

Looking at the above plot, there are a few thing that we notice:

1. Although the signal is of period 2, we have significant energy at period 1 (the dc value) and at period 3. This is due to the non-zero dc component and the fact that a dc signal is also periodic with period 3.

2. The signal has equal energy at period 2 and 4. So what period does the signal have? Is it 2 or 4? If a signal is of period P it will also be of period $2P$, $3P$, etc.

In order to clear up the above ambiguities, we will introduce the following definition.

Definition 1 *We say that a signal S is of* ***exactly period*** *P if the projection of S onto $\mathcal{R}(\Psi^P)$ is non-zero and the projection of S onto $\mathcal{R}(\Psi^{\bar{P}})$ is zero for all $\bar{P} < P$.*

In our example, the received signal R is not **exactly period** 2 since the projection of R onto $\mathcal{R}(\Psi^1)$ is not zero. Similarly, the signal is not **exactly period** 4 or **exactly period** 3. A signal that would be **exactly period** 4 would be

$$R = [-1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1].$$

With our new definition, let $S_1, \ldots, S_m$ be **exactly periodic** with periods $P_1, \ldots, P_m$, respectively. The received signal R, of length K_0 (K_0, a multiple of $P_1, \ldots, P_m$) then consists of $S_1, \ldots, S_m$ plus zero mean, white Gaussian noise N

$$R = S_1 + \ldots + S_m + N$$

With an unknown variance, the ML estimator maximizes the two norm of the sum of the estimates of the sub-signals $\left(\hat{S}_1 + \ldots + \hat{S}_m\right)$, using estimates $\hat{\sigma}^2$ and $\hat{P}_1, \ldots, \hat{P}_m$. One way to obtain the two norm squared of $\left(\hat{S}_1 + \ldots + \hat{S}_m\right)$ is to find orthogonal subspaces corresponding to these signals of **exactly periods** $P_1, \ldots, P_m$. Then, projecting R onto these orthogonal subspaces we obtain estimates $\hat{S}_1, \ldots, \hat{S}_m$. Since the subspaces are orthogonal, so will be $\hat{S}_1, \ldots, \hat{S}_m$ and

$$\|\sum_i^m \hat{S}_i\|^2 = \sum_i^m \|\hat{S}_i\|^2$$

In other words, if we project R onto orthogonal subspaces corresponding to signals of **exactly period** P (with P ranging from 1 to P_{max}), the ML estimator of periods would select the m largest 2 norm projections.

5. ORTHOGONAL SUBSPACE DECOMPOSITION

Next, we will show how to find subspaces corresponding to signals of **exactly period** P (theorem 2). We will also show that these subspaces are indeed orthogonal to each other (theorem 1). We introduce the following definition:

Definition 2 *Define $\Psi^P_{p_1,\ldots,p_m}$, with p_i divisors of P, to be the matrix whose range is the orthogonal complement of $\mathcal{R}\left[\Psi^{p_1} \ldots \Psi^{p_m}\right]$ inside $\mathcal{R}(\Psi^P)$:*

$$\mathcal{R}(\Psi^P_{p_1,\ldots,p_m}) = \mathcal{R}(\Psi^P) \cap \left(\mathcal{R}\left[\Psi^{p_1} \ldots \Psi^{p_m}\right]\right)^\perp$$

Since $\mathcal{R}(\Psi^{p_i}) \subset \mathcal{R}(\Psi^P)$, $\Psi^P_{p_1,\ldots,p_m}$ is not empty. As we will show, if p_i are all the possible divisors of P, including 1, then $\mathcal{R}(\Psi^P_{p_1,\ldots,p_m})$ is the subspace corresponding to signals of **exactly period** P. But first, we introduce the following lemma:

Lemma 1 *Given a signal R of length K_0 (K_0 a multiple of P_1 and P_2), let $\mathcal{R}(\Psi^{P_1})$ be the subspace corresponding to period P_1 and $\mathcal{R}(\Psi^{P_2})$ be the subspace corresponding to period P_2. Also, let $\mathcal{R}(\Psi^{P_3})$ be the subspace corresponding to period P_3, where P_3 is the greatest common divisor of P_1 and P_2. Then $\mathcal{R}(\Psi^{P_3})$ is the intersection of $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$. Moreover, the orthogonal complement of $\mathcal{R}(\Psi^{P_3})$ in $\mathcal{R}(\Psi^{P_1})$, $\mathcal{R}(\Psi^{P_1}_{P_3})$, is orthogonal to the orthogonal complement of $\mathcal{R}(\Psi^{P_3})$ in $\mathcal{R}(\Psi^{P_2})$, $\mathcal{R}(\Psi^{P_2}_{P_3})$. In other words, the three subspaces of Fig. 3 are mutually orthogonal.*

Proof. Assume that we have two periods P_1 and P_2 and that our received signal R is of length K_0 (K_0 a multiple of P_1 and P_2: $K_0 = P_1 M_1 = P_2 M_2$). Using the notation of section (3), $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$ are two subspaces corresponding to signals of period P_1 (and any of the sub-divisors of P_1) and signals of period P_2 (and any of the sub-divisors of P_2), respectively. For the sake of clarity let $P_1 = 4$ and $P_2 = 6$. By definition Ψ^{P_1} and Ψ^{P_2} are

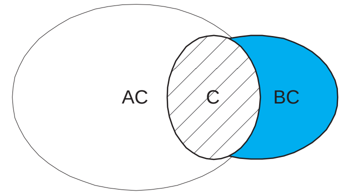

Figure 3: Subspaces $\mathcal{R}(\Psi^{P_1}_{P_3})$, $\mathcal{R}(\Psi^{P_3})$ and $\mathcal{R}(\Psi^{P_2}_{P_3})$ are mutually orthogonal. $\mathcal{R}(\Psi^{P_1}) = \mathcal{R}(\Psi^{P_1}_{P_3}) \oplus \mathcal{R}(\Psi^{P_3})$ and $\mathcal{R}(\Psi^{P_2}) = \mathcal{R}(\Psi^{P_2}_{P_3}) \oplus \mathcal{R}(\Psi^{P_3})$.

$$\Psi^{P_1} = \sqrt{\frac{1}{M_1}} \begin{bmatrix} 1000\ldots \\ 0100\ldots \\ 0010\ldots \\ 0001\ldots \end{bmatrix}^T, \Psi^{P_2} = \sqrt{\frac{1}{M_2}} \begin{bmatrix} 100000\ldots \\ 010000\ldots \\ 001000\ldots \\ 000100\ldots \\ 000010\ldots \\ 000001\ldots \end{bmatrix}^T$$

With P_3, the greatest common divisor of P_1 and P_2, the intersection of $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$ is $\mathcal{R}(\Psi^{P_3})$, as shown in Fig. 3. This is clear since any signal of period P_3 is in both $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$. And any signal that's in both $\mathcal{R}(\Psi^{P_1})$ and $\mathcal{R}(\Psi^{P_2})$ must be of period P_3.

Next, let's find $\mathcal{R}(\Psi^{P_1}_{P_3})$, $\mathcal{R}(\Psi^{P_2}_{P_3})$ and $\mathcal{R}(\Psi^{P_3})$ such that $\mathcal{R}(\Psi^{P_1}) = \mathcal{R}(\Psi^{P_1}_{P_3}) \oplus \mathcal{R}(\Psi^{P_3})$ and $\mathcal{R}(\Psi^{P_2}) = \mathcal{R}(\Psi^{P_2}_{P_3}) \oplus \mathcal{R}(\Psi^{P_3})$. We claim that subspaces $\mathcal{R}(\Psi^{P_1}_{P_3}), \mathcal{R}(\Psi^{P_2}_{P_3})$ and $\mathcal{R}(\Psi^{P_3})$ are

$$\begin{aligned}
\mathcal{R}(\Psi^{P_3}) &= \mathcal{R}(\Psi^{P_1}) \cap \mathcal{R}(\Psi^{P_2}) && (5)\\
&= \mathcal{R}\left[\Psi^{P_1}(\Psi^{P_1})^T \Psi^{P_2}\right] && (6)\\
&= \mathcal{R}\left[\Psi^{P_2}(\Psi^{P_2})^T \Psi^{P_1}\right] && (7)\\
\mathcal{R}(\Psi^{P_1}_{P_3}) &= \mathcal{R}\left[\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T \Psi^{P_1}\right] && (8)\\
\mathcal{R}(\Psi^{P_2}_{P_3}) &= \mathcal{R}\left[\Psi^{P_2} - \Psi^{P_1}(\Psi^{P_1})^T \Psi^{P_2}\right] && (9)
\end{aligned}$$

To prove equations (6) and (7) is a bit long and we will omit the proof here. To prove equations (8) and (9) we have to show that $\mathcal{R}\left[\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T \Psi^{P_1})\right]$ is the orthogonal complement of $\mathcal{R}(\Psi^{P_3})$ in $\mathcal{R}(\Psi^{P_1})$ and that $\mathcal{R}\left[\Psi^{P_2} - \Psi^{P_1}(\Psi^{P_1})^T \Psi^{P_2}\right]$ is the orthogonal complement of $\mathcal{R}(\Psi^{P_3})$ in $\mathcal{R}(\Psi^{P_2})$.

Clearly, any signal in $\mathcal{R}(\Psi^{P_1})$ can be written as a linear combination of vectors in $\mathcal{R}\left[\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T \Psi^{P_1}\right]$ and vectors in $\mathcal{R}(\Psi^{P_3}) = \mathcal{R}\left[\Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1}\right]$; and similarly for $\mathcal{R}(\Psi^{P_2})$.

Next, we need to show:

$$\mathcal{R}\left[\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1}\right] \quad \perp \quad \mathcal{R}(\Psi^{P_3}) \qquad (10)$$
$$\mathcal{R}\left[\Psi^{P_2} - \Psi^{P_1}(\Psi^{P_1})^T\Psi^{P_2}\right] \quad \perp \quad \mathcal{R}(\Psi^{P_3}) \qquad (11)$$

Since $(\Psi^{P_2})^T \times (\Psi^{P_1} - \Psi^{P_2}(\Psi^{P_2})^T\Psi^{P_1}) = 0$ the first orthogonality is proved. Similarly, we have the second orthogonality. We now have that

$$\mathcal{R}(\Psi^{P_1}_{P_3}) \perp \mathcal{R}(\Psi^{P_2}) \quad \Rightarrow \quad \mathcal{R}(\Psi^{P_1}_{P_3}) \perp \mathcal{R}(\Psi^{P_2}_{P_3}) \quad (12)$$

This proves our lemma. Q.E.D.

We are now ready to prove two theorems, which will give us the orthogonality of subspaces corresponding to **exactly period** P.

Theorem 1 *For any two specific periods P and U ($P \neq U$), let $p_1, \ldots, p_n$ and $u_1, \ldots, u_m$ be all the possible divisors of P and U respectively (here we include 1 as a divisor). Then $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$ and $\mathcal{R}(\Psi^U_{u_1,\ldots,u_m})$ are orthogonal.*

Proof.. Without loss of generality, let $p_1 = u_1$ be the greatest common divisor of P and U. Then, $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n}) \subset \mathcal{R}(\Phi^P_{p_1})$ and $\mathcal{R}(\Psi^U_{u_1,\ldots,u_m}) \subset \mathcal{R}(\Psi^U_{u_1})$. By lemma 1, $\mathcal{R}(\Psi^P_{p_1})$ is orthogonal to $\mathcal{R}(\Psi^U_{u_1})$. Q.E.D.

Finally, our last theorem defines the subspace corresponding to signals of **exactly period** P.

Theorem 2 *Let $p_1, \ldots, p_n$ be all the possible divisors of P (here we include 1 as a divisor). If $S \in \mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$, then S is **exactly period** P.*

Proof. For any period $\bar{P} < P$, either $\bar{P}$ is a divisor of P or relatively prime with P. If $\bar{P}$ is a divisor of P, by definition $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$ is orthogonal to $\mathcal{R}(\Psi^{\bar{P}})$. If $\bar{P}$ is relatively prime with P, then their greatest common divisor is one and from theorem 1, $\mathcal{R}(\Psi^P_1)$ is orthogonal to $\mathcal{R}(\Psi^{\bar{P}}_1)$. By definition, $\mathcal{R}(\Psi^P_1)$ is orthogonal to $\mathcal{R}(\Psi^1)$. In other words $\mathcal{R}(\Psi^P_1)$ is orthogonal to $\mathcal{R}(\Psi^{\bar{P}}_1) \oplus \mathcal{R}(\Psi^1) = \mathcal{R}(\Psi^{\bar{P}})$. With $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n}) \subset \mathcal{R}(\Psi^P_1)$, $\mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$ is orthogonal to $\mathcal{R}(\Psi^{\bar{P}})$. In other words, the projection of $S \in \mathcal{R}(\Psi^P_{p_1,\ldots,p_n})$ onto $\mathcal{R}(\Psi^{\bar{P}})$ is zero for all $\bar{P} < P$. Q.E.D.

6. CALCULATION OF THE ORTHOGONAL PROJECTIONS FOR M-PER

In section (3), equation (3) is a fast way of calculating the projection of R onto subspaces $\mathcal{R}(\Psi^P)$. As we stated earlier, in section (4), we would now like to compute the projection of R onto the orthogonal subspaces corresponding to **exactly period** P. In calculating those projections, we can use equation (3), or the slightly modified equation (4), without ever explicitly forming the subspaces corresponding to **exactly period** P. Here is the algorithm using Matlab notation:

```
% Array 'g' is calculated using equation (3)
ExactlyPeriod(1)=g(1);

% Lemma 1 assumes the length of R is a multiple
% of the least common multiplier of P and U.
% That means that for a signal of length L,
% orthogonality of subspaces of exactly
% period P is true only for
% periods P=1 up to approximately P=sqrt(L)
for i=2:sqrt(length(R)),

  % Get all factors, including 1
  fact=all_possible_factors(i);
  fact=[1 fact];

  % Projection of R onto the subspace of period
  % P is equal to the sum of all the projection
  % of R onto subspaces of exactly period P_i
  % (P_i includes all the factors of P, including
  % P and 1)
  ExactlyPeriod(i)=g(i)-sum(ExactlyPeriod(fact));
end
```

Applying the M-PER transform to the signal $R = [1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2]$ of section (4) we obtain the plot of Fig. 4 (top). In this transform, the signal contains a sub-signal of **exactly period** 1 and a sub-signal of **exactly period** 2. There are no sub-signals of **exactly period** 3 or **exactly period** 4. The sub-signal of **exactly period** 1 is the dc component.

Applying the M-PER transform to the synchronized gear box vibration data we obtain the periodic energy shown in Fig. 4 (second from top). Next, we extracted the five largest periodic signals. (These signals correspond to periods: 87, 125, 143, 164 and 238.) In Fig. 4 (bottom), you can also see that the sum of the five largest periodic signals, does a decent job at approximating the original vibration signal.

7. CONCLUSION

We have discussed the application of period and periodic sub-signal estimation as it pertains to machine vibration signals. We would also like to thank Sandip Bose for his valuable discussions on the period estimation problem.

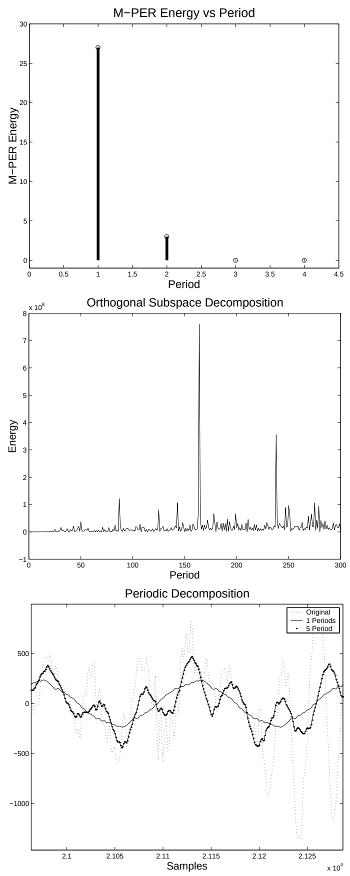

Figure 4: M-PER energy of original signal $R = [1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2]$ (top). M-PER energy of the gear box vibration signal (second from top) and plot of the original vibration signal together with the largest energy periodic signal and the sum of all the five largest periodic signals (bottom).

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